**LECTURE NOTES ON TRIGONOMETRY**

*By*

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**Abstract**

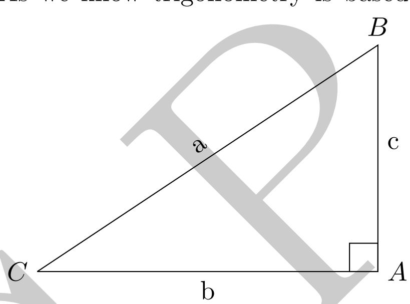
In this lecture note, we give detailed explanation and set of problems related to Trigonometry.

### Topic Covered

Trigonometric ratios of allied, compound, multiple and sub-multiple angles, Factorization and defactorization formula, Inverse Trigonometric ratios, Properties of Triangle.

### 1. Trigonometric Ratios

As we know trigonometry is based upon ratios of the sides of right angle triangle.




In the above $\triangle ABC$, if $\angle ACB = \theta$, then the six trigonometric functions defined by

$$\sin\theta = \frac{c}{b}; \quad \cos\theta = \frac{a}{b}; \quad \tan\theta = \frac{c}{a}$$
$$\cot\theta = \frac{a}{c}; \quad \sec\theta = \frac{b}{a}; \quad \csc\theta = \frac{b}{c}$$

is called trigonometric ratios.

**Exercise 1.** Answer each question about trigonometric ratios with true or false.

1. There are three trigonometric ratios.
2. The sine ratio is the opposite compared to the adjacent.

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3. The tangent ratio is the opposite compared to the adjacent.
4. The sine ratio is the opposite compared to the hypotenuse.
5. The cosine ratio is the adjacent compared to the hypotenuse.

## 2. Fundamental Trigonometrical Identities

An Identity is a mathematical statement that is always true. Here we recall the fundamental identities.

### 2.1. Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

### 2.2. Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

### 2.3. Even-Odd Identities

The cosine and secant functions are even.

$$\cos(-x) = \cos x \qquad \sec(-x) = \sec x$$

The sine, cosecant, tangent, and cotangent functions are odd.

$$\sin(-x) = -\sin x \qquad \csc(-x) = -\csc x$$

$$\tan(-x) = -\cot x \qquad \cot(-x) = -\cot x$$

### 2.4. Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \quad 1 + \tan^2\theta = \sec^2\theta \quad 1 + \cot^2\theta = \csc^2\theta$$

### 2.5. Sum and Difference Identities

$$\begin{aligned}
\sin(x+y) &= \sin x \cos y + \cos x \sin y \\
\sin(x-y) &= \sin x \cos y - \cos x \sin y \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
\cos(x-y) &= \cos x \cos y + \sin x \sin y \\
\tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\
\tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x \tan y}
\end{aligned}$$

### 2.6. Double - Angle Identities

$$\sin 2x = 2 \sin x \cos x$$

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
&= 2\cos^2 x - 1 \\
&= 1 - 2\sin^2 x
\end{aligned}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

### 2.7. Power-Reducing Identities

$$\begin{aligned}
\sin^2 x &= \frac{1 - \cos 2x}{2} \\
\cos^2 x &= \frac{1 + \cos 2x}{2} \\
\tan^2 x &= \frac{1 - \cos 2x}{1 + \cos 2x}
\end{aligned}$$

### 2.8. Half-Angle Identities

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos 2x}{2}}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos 2x}{2}}$$

$$\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

where the sign is determined by the quadrant in which $\frac{x}{2}$ lies.

### 2.9. Product-to-Sum Identities

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$
$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$
$$\cos x \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$$

### 2.10. Sum-to-Product Identities

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

## 3. Trigonometrical Ratios of Allied Angle

First we define allied angle.

**Definition 1.** Two angle are said to be allied if their sum or difference is a multiple of $\frac{\pi}{2}$ radians.

**Example 1.** $-\theta$ is allied to angle $\theta$ as $-\theta + \theta = 0$.

**Example 2.** $90^\circ + \theta$ is allied to angle $\theta$ as $(90^\circ + \theta) - \theta = 90^\circ$.

**Example 3.** $180^\circ + \theta$ is allied to angle $\theta$ as $(180^\circ + \theta) - \theta = 90^\circ \times 2$.

### 3.1. Trigonometric ratios of $-\theta$ in terms of $\theta$

In this section we shall find the trigonometric ratios involving allied angles $\theta$ and $-\theta$.

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$
$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$
$$\tan(-\theta) = -\cot\theta \qquad \cot(-\theta) = -\cot\theta$$

Here we observe the following two points:

1. $\sin(-\theta) = -\sin\theta$, and $\cos(-\theta) = \cos\theta$. These facts follow from the symmetry of the unit circle about the $x$- axis. The angle $-\theta$ is the same as angle $\theta$ except it is on the other side of the $x$- axis. Flipping a point $(x, y)$ to the other side of the $x$- axis makes the point into $(x, y)$, so the $y$-coordinate is negated and hence the sine is negated, but the $x$- coordinate remains the same and therefore the cosine is unchanged.

2. The negative-angle identities can be used to determine if a trigonometric function is an odd function or an even function.

**Exercise 2.** Find the values of $\sin(-45^\circ), \cos(-45^\circ), \tan(-45^\circ)$.

### 3.2. Trigonometric ratios of $(90^\circ - \theta)$, $\left(0 < \theta < \frac{\pi}{2}\right)$ in terms of $\theta$

The trigonometric ratios of angle $(90^\circ - \theta)$ are

$$\sin(90^\circ - \theta) = \cos\theta, \qquad \cos(90^\circ - \theta) = \sin\theta, \qquad \tan(90^\circ - \theta) = \cot\theta$$

$$\csc(90^\circ - \theta) = \sec\theta, \qquad \sec(90^\circ - \theta) = \csc\theta, \qquad \cot(90^\circ - \theta) = \tan\theta$$

### 3.3. Trigonometric ratios of $(90^\circ - \theta)$, $\left(0 < \theta < \frac{\pi}{2}\right)$ in terms of $\theta$

The trigonometric ratios of angle $(90^\circ + \theta)$ are

$$\sin(90^\circ + \theta) = \cos\theta, \qquad \cos(90^\circ + \theta) = -\sin\theta, \qquad \tan(90^\circ + \theta) = -\cot\theta$$

$$\csc(90^\circ + \theta) = \sec\theta, \qquad \sec(90^\circ + \theta) = -\csc\theta, \qquad \cot(90^\circ + \theta) = -\tan\theta$$

The trigonometric function of other allied angles $\pi \pm \theta, \dfrac{3\pi}{2} \pm \theta, \pi \pm \theta$ can be obtained in a similar way.

**Summarization Table**

The above results can be summarized in the following table for $\left(0 < \theta < \frac{\pi}{2}\right)$.

| | -$\theta$ | $\frac{\pi}{2} - \theta$ | $\frac{\pi}{2} + \theta$ | $\pi - \theta$ | $\pi + \theta$ | $\frac{3\pi}{2} - \theta$ | $\frac{3\pi}{2} + \theta$ | $2\pi - \theta$ | $2\pi + \theta$ |
|---|---|---|---|---|---|---|---|---|---|
| sin | $-\sin\theta$ | $\cos\theta$ | $\cos\theta$ | $\sin\theta$ | $-\sin\theta$ | $-\cos\theta$ | $-\cos\theta$ | $-\sin\theta$ | $\sin\theta$ |
| cos | $\cos\theta$ | $\sin\theta$ | $-\sin\theta$ | $-\cos\theta$ | $-\cos\theta$ | $-\sin\theta$ | $\sin\theta$ | $\cos\theta$ | $\cos\theta$ |
| tan | $-\tan\theta$ | $\cot\theta$ | $-\cot\theta$ | $-\tan\theta$ | $\tan\theta$ | $\cot\theta$ | $-\cot\theta$ | $-\tan\theta$ | $\tan\theta$ |

**Points to Remember**

The following points should be remember during finding the trigonometrical ratio of allied angles.

1. The corresponding reciprocal ratios can be written using the above table.

2. If the allied angles are $-\theta, \pi \pm \theta, 2\pi \pm \theta$, that is angle of the form $2n\frac{\pi}{2} \pm \theta$, $n \in \mathbb{Z}$, then, the form of trigonometric ratio is unaltered (i.e., sine remains sine, cosine remains cosine etc.,)

3. If the allied angles are $\frac{\pi}{2} \pm \theta, \frac{3\pi}{2} \pm \theta$, that is, angles of the form $(2n+1)\frac{\pi}{2} \pm \theta$, $n \in \mathbb{Z}$, then the form of trigonometric ratio is altered to its complementary ratio. That means it is to add the prefix "co"if it is absent and remove the prefix "co"if it is already present (i.e., sine becomes cosine, cosine become sine etc.,)

4. For determining the sign, first find out the quadrant and then attach the appropriate sign(+ or-) according to the quadrant rule "ASTC".

**Exercise 3.** Find the value of (i) $\sin 150°$ (ii) $\cos 135°$ (iii) $\tan 120°$.

**Exercise 4.** Find the value of (i) $\sin 765°$ (ii) $\csc(-1410°)$ (iii) $\cot\left(\frac{-15\pi}{4}\right)$.

## 4. Periodicity of Trigonometric Functions

The definition of periodic function is given as follows.

**Definition 2.** A function $f$ is said to be a periodic function with period $p$, if there exists a smallest positive number $p$ such that $f(x+p) = f(x)$ for all $x$ in the domain.

**Example 4.** $\sin(x + 2n\pi) = \sin x, n \in \mathbb{Z}$. That is,

$$\sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) = \cdots = \sin x.$$

Thus, $\sin x$ is a periodic function with period $2\pi$.

**Example 5.** $\cos x$, $\csc x$ and $\sec x$ are periodic functions with period $2\pi$.

**Example 6.** $\tan x$ and $\cot x$ are periodic functions with period $\pi$.

## 5. Odd and Even Trigonometric Functions

The definition of odd and even function is given as follows.

**Definition 3.** A real valued function$f(x)$ is an even function if it satisfies $f(-x) = f(x)$ for all real number $x$ and an odd function if it satisfies $f(-x) = -f(x)$ for all real number $x$.

**Example 7.** $\cos x$ is an even function function as $\cos(-x) = \cos x$ for all $x$.

**Example 8.** $\sin x$ is an odd function function as $\sin(-x) = -\sin x$ for all $x$.

**Exercise 5.** Determine whether the following functions are even, odd or neither.
(a) $\sin^2 x - 2\cos^2 x - \cos x$ (b) $\sin(\cos x)$ (c) $\cos(\sin x)$ (d) $\sin x + \cos x$.

**Points to Remember**

1. In general, a function is an even function if its graph is unchanged under reflection about the $y$-axis. A function is odd if its graph is symmetric about the origin.

2. The properties of even and odd functions are useful in analyzing trigonometric functions particularly in the sum and difference formula.

3. The properties of even and odd functions are useful in evaluating some definite integrals,which we will see in calculus.

## 6. Trigonometrical Ratios of Compound Angles

The definition of compound angle is given as

**Definition 4.** An angle made up of the sum or difference of two or more angles is called a compound angle.

The following is the trigonometric ratios of compound angles.

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \\
\cot(\alpha+\beta) &= \frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha} \\
\cot(\alpha-\beta) &= \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha} \\
\sin(\alpha+\beta)\sin(\alpha-\beta) &= \sin^2\alpha-\sin^2\beta \\
&= \cos^2\beta-\cos^2\alpha \\
\cos(\alpha+\beta)\cos(\alpha-\beta) &= \cos^2\alpha-\sin^2\beta \\
&= \cos^2\beta-\sin^2\alpha
\end{aligned}
$$

## 7. Trigonometrical Ratios of Multiple Angles

Multiple angle is defined as

**Definition 5.** An angle that can be written in the form $nA$, where $n$ is an integer, is called multiple angle.

Now, we shall give trigonometric ratios of multiple angles.

$$\begin{aligned}
\sin 2A &= 2\sin A\cos A \\
&= \frac{2\tan A}{1+\tan^2 A} \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A \\
&= \frac{1-\tan^2 A}{1+\tan^2 A} \\
\tan 2x &= \frac{2\tan A}{1-\tan^2 A} \\
\sin 3A &= 3\sin A - 4\sin^3 A \\
\cos 3A &= 4\cos^3 A - 3\cos A
\end{aligned}$$

## 8. Trigonometrical Ratios of Sub - Multiple Angles

Sub - Multiple angle is defined as

**Definition 6.** An angle that can be written in the form $\frac{A}{n}$, where $n$ is a positive integer, is called sub - multiple angle.

Trigonometric ratios of sub - multiple angles can be easily derived from trigonometric ratios of multiple angles by applying suitable substitution.

## 9. Factorization Formulae

$$\begin{aligned}
\sin C + \sin D &= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \\
\sin C - \sin D &= 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \\
\cos C + \cos D &= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \\
\cos C - \cos D &= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)
\end{aligned}$$

## 10. Defactorization Formulae

$$
\begin{aligned}
2\sin A\sin B &= \cos(A-B)-\cos(A+B)\\
2\cos A\cos B &= \cos(A-B)+\cos(A+B)\\
2\sin A\cos B &= \sin(A+B)+\sin(A-B)\\
2\cos A\sin B &= \sin(A+B)-\sin(A-B)
\end{aligned}
$$

## 11. Inverse Trigonometric Ratios

### 11.1. Inverse of sine Function

As we know that a function has an inverse if and only if it is one-one and onto. Also note that sin is not one-to-one over real numbers. In order to consider the inverse of this function, we need to restrict the domain so that we have a section of the graph that is one-to-one. If the domain of $f$ is restricted to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ a new function $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is defined. This new function is one-to-one and takes on all the values that the function $f(x) = \sin x$ takes on. Since the restricted domain is smaller, $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ takes on all values once and only once. As usual the inverse of $f(x)$ was represented by the symbol $f^{-1}(x)$ and

$$y = f^{-1}(x) \iff f(y) = x.$$

In the same fashion, the inverse of $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ will be denoted as $\sin^{-1} x$ or $\arcsin x$. More precisely,

$$y = \sin^{-1} x \iff \sin y = x.$$

For existence of inverse, the domain of $y = \sin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range is $[-1, 1]$. Hence, the domain of $y = \sin^{-1} x$ is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

### 11.2. Inverse of cosine Function

The inverse function for cosine is defined by following the same process as was applied for the inverse sine function. The domain of $y = \cos^{-1} x$ is $[-1, 1]$ and the range is $[0, \pi]$.

### 11.3. Inverse of tangent Function

The inverse functions for tangent is also defined by following the same process as was applied for the inverse sine function. The domain of $y = \tan^{-1} x$ is $\mathbb{R}$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

### 11.4. Inverse of cotangent Function

The inverse functions for cotangent is also defined by following the same process as was applied for the inverse sine function. The domain of $y = \cot^{-1} x$ is $\mathbb{R}$ and the range is $[0, \pi]$.

### 11.5. Inverse of secant Function

The inverse functions for secant is also defined by following the same process as was applied for the inverse sine function. The domain of $y = \sec^{-1} x$ is $\mathbb{R} - (-1, 1)$ and the range is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

### 11.6. Inverse of co-secant Function

The inverse functions for co-secant is also defined by following the same process as was applied for the inverse sine function. The domain of $y = \csc^{-1} x$ is $\mathbb{R} - (-1, 1)$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

### 11.7. Properties and Relation of Inverse Trogonometric Functions

#### 11.7.1. Self-Adjusting Property

$$\sin(\sin^{-1} x) = x = \sin^{-1}(\sin x)$$
$$\cos(\cos^{-1} x) = x = \cos^{-1}(\cos x)$$
$$\tan(\tan^{-1} x) = x = \tan^{-1}(\tan x)$$

#### 11.7.2. Reciprocal Property

$$\sin^{-1}\left(\frac{1}{x}\right) = \csc^{-1} x$$
$$\sin^{-1}(x) = \csc^{-1}\left(\frac{1}{x}\right)$$

$$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$$
$$\cos^{-1}(x) = \sec^{-1}\left(\frac{1}{x}\right)$$

$$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$$
$$\tan^{-1}(x) = \cot^{-1}\left(\frac{1}{x}\right)$$

### 11.7.3. Conversion Property

$$\begin{aligned}\sin^{-1}(x) &= \cos^{-1}\sqrt{1-x^2}\\ &= \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\\ &= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\end{aligned}$$

$$\begin{aligned}\cos^{-1}(x) &= \sin^{-1}\sqrt{1-x^2}\\ &= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\\ &= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\end{aligned}$$

$$\begin{aligned}\tan^{-1}(x) &= \sec^{-1}\sqrt{1+x^2}\\ &= \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\\ &= \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\end{aligned}$$

## 11.8. Inverse Function of Negative Domain

$$\begin{aligned}\sin^{-1}(-x) &= -\sin^{-1}(x)\\ \cos^{-1}(-x) &= \pi-\cos^{-1}(x)\\ \tan^{-1}(-x) &= -\tan^{-1}(x)\\ \cot^{-1}(-x) &= \pi-\cot^{-1}(x)\\ \sec^{-1}(-x) &= \pi-\sec^{-1}(x)\\ \csc^{-1}(-x) &= -\csc^{-1}(x)\end{aligned}$$

## 11.9. Principal Value of Trigonometric Function

The value of an inverse trigonometric functions which lies in the range is called the principal value of that inverse trigonometric functions.

**Example 9.** Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

**Solution 1.** We have,

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right)$$
$$= -\frac{\pi}{6}$$

Since $-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Hence the principal value will be $-\frac{\pi}{6}$.

**Exercise 6.** Find the principal value of $\tan^{-1}(-\sqrt{3})$.

**Exercise 7.** Find the principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

**Exercise 8.** Find the principal value of $2\cos^{-1}\left(\frac{1}{2}\right) + 3\sin^{-1}\left(\frac{1}{2}\right)$.

## 12. Properties of Triangle

In this section we will discussion about properties of triangle $\triangle ABC$.

### 12.1. Sine Rule

The sides of a triangle are proportional to the sine of the opposite angles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In other words, the sine of the angles of a triangle are proportional to opposite sides.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

**Example 10.** In $\triangle ABC$, $a = 16, b = 12$ and $\angle B = 30°$, find $\sin A$.

**Solution 2.** From sine rule, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\therefore \sin A = \frac{a \sin B}{b}$$
$$= \frac{16 \sin 30°}{12}$$
$$= \frac{16}{24}$$
$$= \frac{2}{3}$$

**Exercise 9.** If the angles of a triangle are in the ratio $3 : 4 : 5$. Find the ratio of its sides.

**12.2. Projection Formulae**

In $\triangle ABC$, the relation between three sides and two angles is given as follows.

$$a = b\cos C + c\cos B$$
$$b = c\cos A + a\cos C$$
$$c = a\cos B + b\cos A$$

**Example 11.** In $\triangle ABC$, prove that

$$(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c.$$

*Proof.* We have,

$$\begin{aligned}
&(b+c)\cos A + (c+a)\cos B + (a+b)\cos C \\
&= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C \\
&= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (b\cos C + c\cos B) \\
&= a+b+c. \qquad \text{(by projection formulae)}
\end{aligned}$$

□

**12.3. Law of Tangents**

In $\triangle ABC$, the following properties also holds.

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$
$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$
$$\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$$

***The End***