LECTURE NOTES ON BINOMIAL THEOREM

By

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Abstract

In this lecture note, we give detailed explanation and set of problems related to Binomial theorem for negative index.

Topic Covered: Binomial theorem for negative index, Approximate value (only formula)

1. Binomial Theorem for Negative Index

Theorem 1. *If n is a negative integer and x is a real number with $| x |< 1$, then*

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r+\cdots$$

1.1. Remarks

- The above statement also holds for positive integers. Thus, for a positive integer n, we have

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n!}x^n.$$

- The general term in above expansion is

$$T_{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$$

- In the expansion like $(a+x)^n$, the first term should be made unity.

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1.2. Particular Cases

- Replacing n by $-n$ and simplifying, we get

$$(1+x)^{-n} = 1-nx+\frac{n(n+1)}{2!}x^2-\cdots+(-1)^{r-1}\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r+\cdots$$

- Replacing x by $-x$ and simplifying, we get

$$(1-x)^n = 1-nx+\frac{n(n-1)}{2!}x^2-\cdots+(-1)^r\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r+\cdots$$

- Replacing n by $-n$ and replacing x by $-x$ and simplifying, we get

$$(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\cdots+\frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}x^r+\cdots$$

Example 1. Expand $(1-2x^{-1})$ up to 5 terms.

Solution 2. By applying the Binomial theorem for negative index, we have

$$\begin{aligned}(1-2x)^{-1} &= 1+(2x)+(2x)^2+(2x)^3+(2x)^4+\cdots \\ &= 1+2x+4x^2+8x^3+16x^4 \quad \text{(up to 5 terms)}\end{aligned}$$

Example 2. Find the coefficient of x^3 in the expansion of $\dfrac{(1+3x)^2}{1-2x}$.

Solution 3. We have,

$$\begin{aligned}\frac{(1+3x)^2}{1-2x} &= (1+3x)^2(1-2x)^{-1} \\ &= (1+6x+9x^2)(1+2x+4x^2+8x^3+\cdots).\end{aligned}$$

Hence, the coefficient of x^3

$$\begin{aligned}&= (1)(8)+(6)(4)+(9)(2) \\ &= 8+24+18 \\ &= 50.\end{aligned}$$

Exercise 1. Find the coefficient of x^7 in the expansion of $(x - 2x^2)^3$.

Exercise 2. Find the negative value of n, if the coefficient of x^2 in the expansion of $(1 + x)^m$ is 6.

Exercise 3. Find the coefficient of x^n in the expansion of $\dfrac{(1+x)^2}{(1-x)^3}$. Also find the coefficients of x^5 and x^7.

2. Approximate Value

Approximating powers of numbers by using Binomial theorem is called approximate value. The reason behind this fact is that if x is sufficiently small then x^2 and higher powers of x can be neglected and as a result, we get approximate value up to two terms

$$(1 + x)^n \approx 1 + nx.$$

Similarly, in the same fashion, the approximate value up to three terms

$$(1 + x)^n \approx 1 + nx + \frac{n(n-1)}{2!}x^2.$$

Example 3. Indicate which is larger by using Binomial theorem

$$(1.1)^{10000} \quad \text{or} \quad 1000.$$

Solution 4. Using Binomial theorem, we have

$$\begin{aligned}
(1.1)^{10000} &= (1 + 0.1)^{1000} \\
&= 1 + \binom{10000}{1}(1)^{9999}(0.1) + \text{other positive terms.} \\
&= 1 + 10000\ (0.1) + \text{other positive terms} \\
&= 1 + 1000 + \text{other positive terms.}
\end{aligned}$$

Thus, $(1.1)^{10000} > 1000$.

Exercise 4. Indicate which is larger by using Binomial theorem

$$(1.2)^{4000} \quad \text{or} \quad 800.$$

Exercise 5. Using Binomial theorem, evaluate each of the following:

1. 99^5
2. 101^4
3. 96^3

Exercise 6. Find cube root of 998 correct upto 5 decimal places.

***The End ***