LECTURE NOTES ON BINOMIAL THEOREM

By

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Abstract

In this lecture note, we give detailed explanation and set of problems related to Binomial theorem.

Topic Covered: Binomial theorem for positive index. General and Middle term(s) of the Binomial Expansion.

1. Useful Definition

Before presenting the Binomial theorem, we need to define Binomial expression.

Definition 1. A two terms algebraic expression is called binomial expression.

Example 1. $x+7$, $x+2a$, etc.

1.1. Binomial Theorem

Theorem 1. *If n is a positive integer, then*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}y^n.$$

In other words,

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r.$$

Remarks:

- The coefficients $\binom{n}{r}$ occuring in the binomial theorem are known as binomial coefficients.
- There are $n+1$ terms in the expansion of $(x+y)^n$.
- The number n is called index of the binomial.
- The n^{th} number of term is denoted by T_n.

Example 2. Find the number of terms in the expansions of $[(2x+3y)^2]^5$.

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Solution 2. We have,

$$[(2x+3y)^2]^5 = (2x+3y)^{10}$$

Here, Index of the binomial is $n = 10$. Thus the number of terms in the expansion $= n+1$=10+1=11.

Example 3. Find the number of terms in the expansions of $[(3x+y^2)^9]^4$.

Solution 3.

We have,

$$[(3x+y^2)^9]^4 = (3x+y^2)^{36}$$

Here, Index of the binomial is $n = 36$. Thus the number of terms in the expansion $= n+1$=36+1=37.

Exercise 1. Find the number of terms in the following expansions:

1. $(x+3)^8$
2. $(a-2b)^{12}$

2. General Term of the Binomial Expansion

The $(r+1)^{\text{th}}$ in the expansion of $(x+y)^n$ is called general term of the binomial expansion. It is denoted by T_{r+1} and is defined as

$$T_{r+1} = \binom{n}{r} x^{n-r} y^r.$$

Example 4. Find the general term in the expansions of $(x+3)^8$.

Solution 4. The given expression is $(x+3)^8$. Hence, the general term of given expression is

$$T_{r+1} = \binom{8}{r} x^{8-r} 3^r.$$

Exercise 2. Find the general term in the following expansions:

1. $(a-2b)^{12}$
2. $(x^2-yx)^{12}$, when $x \neq 0$.

3. Middle Term(s) of the Binomial Expansion

The number of terms in the binomial expansion of $(x+y)^n$ is $n+1$. There are two cases arises for finding middle term(s):

Case-I: When n is even
In this case the number of terms in the binomial expansion is odd. So, there is only one middle term in the expansion, namely,

$$\left(\frac{n+2}{2}\right)^{\text{th}} \text{term.}$$

Case-II: When n is Odd
In this case the number of terms in the binomial expansion is even. So, there are two middle terms in the expansion, namely,

$$\left(\frac{n+1}{2}\right)^{\text{th}} \text{term} \quad \text{and} \quad \left(\frac{n+3}{2}\right)^{\text{th}} \text{term.}$$

Example 5. Find the middle term(s) in the expansions of $(\frac{x}{3}+9y)^{10}$.

Solution 5. The given expression is $(\frac{x}{3}+9y)^{10}$. Here, $n=10$, which is even. So, the middle term of the expansion is

$$\left(\frac{n+1}{2}\right)^{\text{th}} \text{term} = 6^{\text{th}}\text{term.}$$

For 6^{th} term, we have

$$r+1=6 \implies r=5.$$

Hence, the middle term is

$$\begin{aligned} T_6 &= \binom{10}{5}\left(\frac{x}{3}\right)^{10-5}(9y)^5 \\ &= \binom{10}{5}\left(\frac{x}{3}\right)^{5}(9y)^5 \\ &= 61236 \quad x^5y^5. \end{aligned}$$

Exercise 3. Find the middle term(s) in the following expansions:

1. $\left(3x-\frac{x^3}{6}\right)^7$
2. $\left(\frac{p}{x}+\frac{x}{p}\right)^9$.

4. Miscellaneous Exercise

1. Find the cofficient of x^5 in the expansion of $(x+3)^8$.
2. Find the value of a if the 17^{th} and 18^{th} terms of the expansion $(2+a)^{50}$ are equal.

3. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, find the value of p.

4. Find the term independent of x in the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^6$.

5. Which term in the expansion of $\left(x^3+\frac{2}{x^2}\right)^{15}$ is independent of x? Also find its value.

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