LECTURE NOTES ON DERIVATIVES

By

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Abstract

In this lecture note, we give detailed explanation and set of problems on derivatives.

Topic Covered: Derivatives of standard functions, Derivatives of trigonometric functions, Derivatives of composite functions (Chain rule), Derivatives of exponential and logarithmic functions

1. Some Useful Results

In this section, we give some useful results on derivatives.

Theorem 1. $f^{'}(x) = \lim_{t \to x} \dfrac{f(t) - f(x)}{t - x}$.

Proof. From the definition of derivative,

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Let $x + h = t$. Then $t \to x$ as $h \to 0$. Thus,

$$f^{'}(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.$$

□

Theorem 2. *The derivative of a constant function is zero.*

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Proof. Let $f(x) = c$, for all $x \in \mathbb{R}$. Then by definition

$$\begin{aligned} f^{'}(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= 0. \end{aligned}$$

Thus,

$$\frac{d}{dx} f(x) = 0$$

. □

Theorem 3. *For any constant $k \in \mathbb{R}$,*

$$\frac{d}{dx} k f(x) = k \frac{d}{dx} f(x).$$

Proof. From the definition,

$$\begin{aligned} \frac{d}{dx} k f(x) &= \lim_{h \to 0} \frac{k f(x+h) - k f(x)}{h} \\ &= \lim_{h \to 0} k \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(using product rule of limit)} \\ &= k \frac{d}{dx} f(x). \end{aligned}$$

Thus,

$$\frac{d}{dx} k f(x) = k \frac{d}{dx} f(x).$$

□

2. Derivatives of Standard Functions

In this section, we give the derivative of some standard functions.

Theorem 4. *For any natural number n,* $\frac{d}{dx} x^n = n x^{n-1}$.

2.1. Derivatives of Trigonometric Functions

- Derivative of sine function:

$$\frac{d}{dx}\sin x = \cos x.$$

- Derivative of cosine function:

$$\frac{d}{dx}\cos x = -\sin x.$$

- Derivative of tangent function:

$$\frac{d}{dx}\tan x = -\sec^2 x.$$

- Derivative of cotangent function:

$$\frac{d}{dx}\cot x = -\csc^2 x.$$

- Derivative of secant function:

$$\frac{d}{dx}\sec x = -\sec x \tan x.$$

- Derivative of cosecant function:

$$\frac{d}{dx}\csc x = -\csc x \cot x.$$

3. Derivative of Composite Functions

The most useful method for finding derivative of composite functions is the Chain rule. This rule allows us to differentiate complicated functions in terms of known derivatives of simpler functions.

Theorem 5 (The Chain Rule)**.** *If g is a differentiable function at x and f is differentiable at $g(x)$, then the composition function $f \circ g = f(g(x))$ is differentiable at x. The derivative of the composite function is:*

$$f \circ g = f^{'}(g(x))g^{'}(x).$$

Another way of expressing, *if $u = u(x)$ and $f = f(u)$, then*

$$\frac{d}{dx} f(u) = f^{'}(u) \frac{du}{dx}.$$

*And **a final way of expressing** the chain rule is the easiest form to remember, if y is a function of u and u is a function of x, then*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 1. Differentiate $f(x) = (2x^3 - 4x^2 + 5)^2$.

Solution 1. Let $u = 2x^3 - 4x^2 + 5$. Then

$$\begin{aligned} \frac{d}{dx}(2x^3 - 4x^2 + 5)^2 &= \frac{d}{dx} u^2 \\ &= 2u \frac{du}{dx} \\ &= 2(2x^3 - 4x^2 + 5)(6x^2 - 8x). \end{aligned}$$

Exercise 1. Find $\dfrac{dy}{dx}$ for $y = \sin^3 x$.

Exercise 2. Find $\dfrac{dy}{dx}$ for $y = 5\cos(3x^2 - 1)$.

3.1. Derivatives of Exponential and Logarithmic Functions

First we recall definition of exponential function.

Definition 1. The function $f(x) = e^x$ is called the natural exponential function. Its inverse, $g(x) = \log_e x = \ln x$ is called the natural logarithmic function.

Definition 2. A function $f(x) = a^x$, $a \neq 1$ and $a > 0$, for all $x \in \mathbb{R}$ is called a general exponential function and for $x, a > 0$ and $a \neq 1$, a function $f(x)$ defined by $f(x) = \log_a x$ is called the general logarithmic function..

3.1.1. Derivatives of the Natural Exponential Function

Theorem 6. *Let $f(x) = e^x$ be the natural exponential function. Then*

$$f^{'}(x) = e^x.$$

In general,

$$\frac{d}{dx}e^{f(x)} = e^{f(x)}f^{'}(x).$$

3.1.2. Derivatives of the Natural Logarithmic Function

Theorem 7. *Let $y = \ln x, x > 0$ be the natural logarithmic function. Then*

$$\frac{dy}{dx} = \frac{1}{x}.$$

In general, for all values of x for which $f(x) > 0$

$$\frac{d}{dx}\left[\ln(f(x))\right] = \frac{1}{f(x)}f^{'}(x).$$

3.1.3. Derivatives of the General Exponential Function

Theorem 8. *Let $y = a^x$ with $a > 0$ and $a \neq 1$. Then*

$$\frac{dy}{dx} = a^x \ln a.$$

In general, if $h(x) = a^{f(x)}$, then

$$h^{'}(x) = a^{f(x)}f^{'}(x)\ln a.$$

3.1.4. Derivatives of the General Logarithmic Function

Theorem 9. *Let $y = \log_a x$ with $a, x > 0$ and $a \neq 1$. Then*

$$\frac{dy}{dx} = \frac{1}{x \ln a}.$$

In general, if $h(x) = \log_a(f(x))$, then for all x for which $f(x) > 0$,

$$h^{'}(x) = \frac{f^{'}(x)}{f(x)\log(a)}.$$

3.2. Problem-Solving Method For Logarithmic Differentiation

To find out the derivatives of the functions such as $y = x^x, x^\pi$ etc., or more generally, $h(x) = g(x)^{f(x)}$, a special technique used what is called **logarithmic differentiation**. The outline of this technique is as follows:

- To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
- Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
- Differentiate both sides of the equation. On the left we will have $\dfrac{1}{y}\dfrac{dy}{dx}$.
- Multiply both sides of the equation by y to solve for $\dfrac{dy}{dx}$.
- Replace y by $h(x)$.

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