LECTURE NOTES ON DERIVATIVES

By

Mritunjay Kumar Singh [1]

Abstract

In this lecture note, we give detailed explanation and set of problems on derivatives.

Topic Covered: Definition and example of derivative, Algebra of derivatives of functions,

1. Motivation

The term derivative introduced in order to know a certain parameter at various instants of time and finding the rate at which it is changing.

Definition 1. A real valued function $f(x)$ is said to be differentiable at a point c in its domain if the

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \quad \textbf{or} \quad \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

exists and is finite. It is denoted by $f^{'}(c)$.

The process of finding derivative is called differentiation.

Example 1. Find the derivative of $\sin x$ at $x = 0$.

Solution 1. Let $f(x) = \sin x$. Then

$$\begin{aligned} f^{'}(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{\sin(0+h)-\sin(0)}{h} \\ &= \lim_{h\to 0} \frac{\sin h}{h} \\ &= 1. \end{aligned}$$

[1] Lecturer in Mathematics, Government Polytechnic, Gaya, Bihar, India, Mobile : 9546595789

Exercise 1. Find the derivative of $f(x) = 3$ at $x = 0$ and $x = 3$.

If the derivative exists at every points of its domain then the derivative of a function is defined as follows.

Definition 2. A real valued function $f(x)$ is said to be differentiable in its domain if the

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is finite. It is denoted by $f^{'}(x)$.

The derivative obtained by above method is called "derivative by first principle". Throughout this lecture it will be assumed that the given function is differentiable at all points in its domain.

1.1. Physical Meaning of $\boldsymbol{\frac{dy}{dx}}$

The derivative of y with respect to x is denoted by $\frac{dy}{dx}$. In other words,

$$\frac{dy}{dx} = \text{Rate of change of } y \text{ with respect to } x.$$

In particular,

$$\begin{aligned}\left(\frac{dy}{dx}\right)_{x=c} &= \text{The value and of } \frac{dy}{dx} \text{ at } x = c.\\ &= \text{Rate of change of } y \text{ with respect to } x \text{ at } x = c.\end{aligned}$$

1.2. Geometrical Meaning of $\boldsymbol{\frac{dy}{dx}}$

Geometrically, $\dfrac{dy}{dx}=$ represents the slope of the tangent to the curve $y = f(x)$ at a point (x, y). In particular,

$$\left(\frac{dy}{dx}\right)_{x=c} = \text{Slope of the tangent to the curve } y = f(x) \text{ at the point } (c, f(c)).$$

If the tangent at a point of the curve $y = f(x)$ makes an angle ψ with positive direction of x-axis, then

$$\frac{dy}{dx} = \tan\psi.$$

Example 2. Find the differential co-efficient of $\cot x$ with respect to x by first principle.

Solution 2. Let $f(x) = \cot x$. Then

$$\begin{aligned}
\frac{df(x)}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\cot(x+h)-\cot(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right] \\
&= \lim_{h\to 0}\left[\frac{\cos(x+h)\sin x - \sin(x+h)\cos x}{h\sin(x+h)\sin x}\right] \\
&= \lim_{h\to 0}\frac{\sin(x-(x+h))}{h\sin(x+h)\sin x} \quad \text{(by using formula of } \sin(A-B)) \\
&= \lim_{h\to 0}\frac{-\sin h}{h\sin(x+h)\sin x} \\
&= -\lim_{h\to 0}\frac{\sin h}{h}\lim_{h\to 0}\frac{1}{\sin(x+h)\sin x} \quad \text{(by using product rule of limits)} \\
&= \frac{-1}{\sin^2 x}.
\end{aligned}$$

Exercise 2. Find the differential co-efficient of $\tan x$ with respect to x by first principle.

Exercise 3. Find the differential co-efficient of $\sin x$ with respect to x by first principle.

2. Algebra of Derivatives of Functions

Let $f(x)$ and $g(x)$ be two functions such that their derivatives are defined in a common domain. Then

1. Sum rule of derivative of functions:
$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x).$$
2. Difference rule of derivative of functions:
$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x).$$
3. product rule of derivative of functions:
$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x).$$
4. quotient rule of derivative of functions:
For non-zero denominator,
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{\left(g(x)\right)^2}.$$

Example 3. Find the derivative of the function $3x^7-5x^2+9$.

Solution 3. Let $y=3x^7-5x^2+9$.
Differentiating both side with respect to x, we have

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(3x^7-5x^2+9\right)\\&=\frac{d}{dx}\left(3x^7\right)+\frac{d}{dx}\left(-5x^2\right)+\frac{d}{dx}(9)\text{ (using sum rule)}\\&=3\frac{d}{dx}\left(x^7\right)+(-5)\frac{d}{dx}\left(x^2\right)+0\\&=3\cdot 7x^6-5\cdot 2x^3\\&=21x^6-10x.\end{aligned}$$

Exercise 4. Using sum rule, find the derivative of the following functions:

1. $(x^2+1)(x-2)$.

2. $\left(x - \frac{1}{x}\right)^3$.

Example 4. Using product rule, find the derivative of the function $x^3 \sin x$.

Solution 4. Let $y = x^3 \sin x$.
Differentiating both side with respect to x, we have

$$\begin{aligned}\frac{dy}{dx} &= x^3 \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x^3 \text{ (using product rule)} \\ &= x^3 \cos x + \sin x (3x^2) \\ &= x^3 \cos x + 3x^2 \sin x.\end{aligned}$$

Exercise 5. Using product rule, find the derivative of the following functions:

1. $(2x^3 - 7)(9x^5 + 2x^2 - 3)$.

2. $(x + \cos x)(x - \tan x)$.

Example 5. Using quotient rule, find the derivative of the function $\dfrac{4x^2 - 1}{(5 - 2x)^3}$.

Solution 5. Let $y = \frac{4x^2-1}{(5-2x)^3}$.
Differentiating both side with respect to x, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{(5-2x)^3 \frac{d}{dx}(4x^2 - 1) - (4x^2 - 1)\frac{d}{dx}(5-2x)^3}{\left[(5-2x)^3\right]^2} \text{ (using quotient rule)} \\ &= \frac{(5-2x)^3 (4 \cdot 2x) - (4x^2 - 1) 3 \cdot (5-2x)^2 \cdot (-2)}{(5-2x)^6} \\ &= \frac{(5-2x)^2 \left[(5-2x) \cdot 8x + 6(4x^2 - 1)\right]}{(5-2x)^6} \\ &= \frac{8x^2 + 40x - 6}{(5-2x)^6}.\end{aligned}$$

Exercise 6. Using quotient rule, find the derivative of the following functions:

1. $\dfrac{3x-2}{5x^2+7}$.
2. $\dfrac{\cos x}{1+\sin x}$.

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